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Budget equations and astrophysical nonlinear mean-field dynamos

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ABSTRACT
Solar, stellar and galactic large-scale magnetic fields are originated due to a combined action of non-uniform (differential) rotation and helical motions of plasma via mean-field dynamos. Usually, nonlinear mean-field dynamo theories take into account algebraic and dynamic quenching of alpha effect and algebraic quenching of turbulent magnetic diffusivity. However, these theories do not take into account a feedback of the mean magnetic field on the background turbulence (with a zero mean magnetic field). Our analysis using the budget equation for the total (kinetic plus magnetic) turbulent energy, which takes into account the feedback of the generated mean magnetic field on the background turbulence, has shown that a nonlinear dynamo number decreases with increase of the mean magnetic field for a forced turbulence, and a shear-produced turbulence and a convective turbulence. This implies that mean-field dynamo instability is always saturated.

Key words: dynamo – MHD – Sun: interior — turbulence – activity – dynamo– galaxies: magnetic fields

1 INTRODUCTION

Large-scale magnetic fields in the sun, stars and galaxies are believed to be generated by a joint action of a differential rotation and helical motions of plasma (see, e.g., Moffatt 1978; Parker 1979; Krause & Rädler 1980; Zeldovich et al. 1983; Ruzmaikin et al. 1988; Rüdiger et al. 2013; Moffatt & Dormy 2019; Rogachevskii 2021; Shukurov & Subramanian 2021). This mechanism can be described by the $\alpha\Omega$ or $\alpha^2\Omega$ mean-field dynamos. In particular, the effect of turbulence in the mean-field induction equation is determined by the turbulent electromotive force, $\langle \boldsymbol{u} \times \boldsymbol{b} \rangle$, which can be written for a weak mean magnetic field $\overline{\boldsymbol{B}}$ as $\langle \boldsymbol{u} \times \boldsymbol{b} \rangle = \alpha_{\rm K} \overline{\boldsymbol{B}} + \boldsymbol{V}^{\rm (eff)} \times \overline{\boldsymbol{B}} - \eta_{_T} (\boldsymbol{\nabla} \times \overline{\boldsymbol{B}})$, where $\alpha_{\rm K}$ is the kinetic α effect caused by helical motions of plasma, $\eta_{_T}$ is the turbulent magnetic diffusion coefficient, $\boldsymbol{V}^{\rm (eff)}$ is the effective pumping velocity caused by an inhomogeneity of turbulence. Here the angular brackets imply ensemble averaging, $\boldsymbol{u}$ and $\boldsymbol{b}$ are fluctuations of velocity and magnetic fields, respectively. The threshold of the $\alpha\Omega$ mean-field dynamo instability is described in terms of a dynamo number $D_{\rm L} = \alpha_{\rm K}\, \delta\Omega\, L^3/\eta_{_T}^2$, where $\delta\Omega$ characterises the non-uniform (differential) rotation and L is the stellar radius or the thickness of the galactic disk.

The mean-field dynamos are saturated by nonlinear effects. In particular, a feedback of the growing large-scale magnetic field on plasma motions is described by algebraic quenching of the α effect, turbulent magnetic diffusion, and the effective pumping velocity. This implies that the turbulent transport coefficients, $\alpha_{\rm K}\left(\overline{B}\right)$, $\eta_{_T}\left(\overline{B}\right)$ and $\boldsymbol{V}^{\rm (eff)}\left(\overline{B}\right)$ depend on the mean magnetic field $\overline{\boldsymbol{B}}$ via algebraic decreasing functions. The quantitative theories of the algebraic nonlinearities of the α effect, the turbulent magnetic diffusion and the effective pumping velocity have been developed using the quasi-linear approach for small fluid and magnetic Reynolds numbers (Rüdiger & Kichatinov 1993; Kitchatinov et al. 1994; Rüdiger et al. 2013) and the tau approach for large fluid and magnetic Reynolds numbers (Field et al. 1999; Rogachevskii & Kleeorin 2000, 2001, 2004, 2006).

In addition to the algebraic nonlinearity, there is also a dynamic nonlinearity caused by an evolution of magnetic helicity density of a small-scale turbulent magnetic field during the nonlinear stage of the mean-field dynamo. In particular, the α effect has contributions from the kinetic α effect, $\alpha_{\rm K}$, determined by the kinetic helicity and a magnetic α effect, $\alpha_{\rm M}$, described by the current helicity of the small-scale turbulent magnetic field (Pouquet et al. 1976). The dynamics of the current helicity are determined by the evolution of the small-scale magnetic helicity density $H_{\rm m} = \langle \boldsymbol{a} {\cdot} \boldsymbol{b} \rangle$, where $\boldsymbol{b} = \boldsymbol{\nabla} {\times} \boldsymbol{a}$ and $\boldsymbol{a}$ are fluctuations of the magnetic vector potential. The total magnetic helicity, i.e., the sum of the magnetic helicity densities of the large-scale and small-scale magnetic fields, $H_{\rm M}+H_{\rm m}$, integrated over the volume, $\int (H_{\rm M} + H_{\rm m})\, dr^3$, is conserved for very small microscopic magnetic diffusivity η. Here $H_{\rm M} = \overline{\boldsymbol{A}}{\cdot}\overline{\boldsymbol{B}}$ is the magnetic helicity density of the large-scale magnetic field $\overline{\boldsymbol{B}} = \boldsymbol{\nabla}{\times}\overline{\boldsymbol{A}}$ and $\overline{\boldsymbol{A}}$ is the mean magnetic vector potential.

As the mean-field dynamo amplifies the mean magnetic field, the large-scale magnetic helicity density $H_{\rm M}$ grows in time. Since the total magnetic helicity $\int (H_{\rm M} + H_{\rm m})\, dr^3$ is conserved for very small magnetic diffusivity, the magnetic helicity density $H_{\rm m}$ of the small-scale field changes during the dynamo action, and its evolution is determined by the dynamic equation (Kleeorin & Ruzmaikin 1982; Zeldovich et al. 1983; Gruzinov & Diamond 1994; Kleeorin et al. 1995; Kleeorin & Rogachevskii 1999).

In a nonlinear $\alpha\Omega$ dynamo one can define a nonlinear dynamo number $D_{\rm N}\left(\overline{B}\right) = \alpha\left(\overline{B}\right)\, \delta\Omega\, L^3/\eta_{_T}^2\left(\overline{B}\right)$. If the nonlinear dynamo number $D_{\rm N}\left(\overline{B}\right)$ decreases with the increase of the large-scale magnetic field, the mean-field dynamo instability is saturated by the nonlinear effects. However, if the α effect and the turbulent magnetic diffusion are quenched as $(\overline{B}/\overline{B}_{\rm eq})^{-2}$ for strong mean magnetic fields, the nonlinear dynamo number $D_{\rm N}\left(\overline{B}\right) \propto (\overline{B}/\overline{B}_{\rm eq})^2$ increases with the increase of the large-scale magnetic field, and the mean-field dynamo instability cannot be saturated for a strong mean magnetic field. Here $\overline{B}_{\rm eq} = \left(\mu_0\, \overline{\rho}\, \langle \boldsymbol{u}^2\rangle\right)^{1/2}$ is the equipartition mean magnetic field and μ_0 is the magnetic permeability of the fluid. How is it possible to resolve this paradox?

The mean-field dynamo theories imply that there is a background helical turbulence with a zero mean magnetic field. Due to the combined effect of the differential rotation and helical motions in the background turbulence (described by the kinetic α effect), a large-scale magnetic field is amplified by the mean-field dynamo instability. In a nonlinear dynamo stage, there is an additional feedback effect of the growing large-scale magnetic field on the background turbulence. However, this effect has not been yet taken into account in nonlinear mean-field dynamo theories.

In the present study, we have taken into account the feedback of the mean magnetic field on the background turbulence using the budget equation for the total (kinetic plus magnetic) turbulent energy. Considering three different types of astrophysical turbulence:

- a forced turbulence (e.g., caused by supernova explosions in galaxies);
- a shear-produced turbulence (e.g., in the atmosphere of the Earth or other planets) and
- a convective turbulence (e.g., in a solar and stellar convective zones),

we have demonstrated that the nonlinear dynamo number decreases for any strong values of the mean magnetic field for these three kinds of turbulence, resulting in saturation of the mean-field dynamo instability.

2 BUDGET EQUATIONS

Using the Navier-Stokes equation for velocity fluctuations, we derive the budget equation for the density of turbulent kinetic energy (TKE), $E_{\rm K} = \overline{\rho}\, \langle \boldsymbol{u}^2\rangle/2$ as

$$\frac{\partial E_{\rm K}}{\partial t} + {\rm div}\, \boldsymbol{\Phi}_{\rm K} = \Pi_{\rm K} - \varepsilon_{\rm K}, \tag{1}$$

where $\boldsymbol{\Phi}_{\rm K} = \left\langle \boldsymbol{u}\left(\rho\, \boldsymbol{u}^2/2 + p\right)\right\rangle - \nu\, \overline{\rho}\, \boldsymbol{\nabla} E_{\rm K}$ is the flux of TKE, $\varepsilon_{\rm K} = \nu\, \overline{\rho}\, \left\langle (\nabla_j u_i)^2\right\rangle$ is the dissipation rate of TKE, and

$$\begin{aligned}\Pi_{\rm K} = &-\frac{1}{\mu_0}\Big[\langle \boldsymbol{u}\cdot[\boldsymbol{b}\times(\boldsymbol{\nabla}\times\boldsymbol{b})]\rangle - \langle \boldsymbol{u}\times(\boldsymbol{\nabla}\times\boldsymbol{b})\rangle\cdot\overline{\boldsymbol{B}}\\ &+ \langle \boldsymbol{u}\times\boldsymbol{b}\rangle\cdot(\boldsymbol{\nabla}\times\overline{\boldsymbol{B}})\Big] + \overline{\rho}\,\Big[g\, F_z - \langle u_i u_j\rangle\, \nabla_j \overline{U}_i\\ &+ \langle \boldsymbol{u}\cdot\boldsymbol{f}\rangle\Big]\end{aligned} \tag{2}$$

is the production rate of TKE. Here $\overline{\boldsymbol{U}}$ is the mean velocity, ν is the kinematic viscosity and the angular brackets imply ensemble averaging, $\boldsymbol{F} = \langle s\, \boldsymbol{u}\rangle$ is the turbulent flux of the entropy, $s = \theta/\overline{T} + (\gamma^{-1} - 1)p/\overline{P}$ are entropy fluctuations, θ and $\overline{T}$ are fluctuations and mean fluid temperature, ρ and $\overline{\rho}$ are fluctuations and mean fluid density, p and $\overline{P}$ are fluctuations and mean fluid pressure, $\gamma = c_{\rm p}/c_{\rm v}$ is the ratio of specific heats, $\boldsymbol{g}$ is the acceleration due to the gravity and $\overline{\rho}\, \boldsymbol{f}$ is the external steering force with a zero mean.

We consider three different cases when turbulence is produced either by convection, or by large-scale shear motions or by an external steering force, see the last three terms in the RHS of Eq. (2). The first two terms in the RHS of Eq. (2) describe an energy exchange between the turbulent kinetic and magnetic energies (see below), and the third term in the RHS of Eq. (2) are due to the work of the Lorentz force in a nonuniform mean magnetic field. The estimate for the dissipation rate of the turbulent kinetic energy density in homogeneous isotropic and incompressible turbulence with a Kolmogorov spectrum is $\varepsilon_{\rm K} = E_{\rm K}/\tau_0$, where τ_0 is the characteristic turbulent time at the integral scale.

Using the induction equation for magnetic fluctuations, we derive the budget equation for the density of turbulent magnetic energy (TME), $E_{\rm M} = \langle \boldsymbol{b}^2\rangle/2\mu_0$ as

$$\frac{\partial E_{\rm M}}{\partial t} + {\rm div}\, \boldsymbol{\Phi}_{\rm M} = \Pi_{\rm M} - \varepsilon_{\rm M}, \tag{3}$$

where

$$\begin{aligned}\boldsymbol{\Phi}_{\rm M} = &\frac{1}{\mu_0}\Big[\langle \boldsymbol{b}\times(\boldsymbol{u}\times\boldsymbol{b})\rangle + \langle \boldsymbol{u}\, b_j\rangle\, \overline{B}_j - \langle \boldsymbol{u}\cdot\boldsymbol{b}\rangle\, \overline{\boldsymbol{B}}\\ &+ \left\langle \boldsymbol{b}^2\right\rangle \overline{\boldsymbol{U}} - \langle \boldsymbol{b}\, b_j\rangle\, \overline{U}_j - \eta\, \langle \boldsymbol{b}\times(\boldsymbol{\nabla}\times\boldsymbol{b})\rangle\Big]\end{aligned} \tag{4}$$

is the flux of TME, $\varepsilon_{\rm M} = \eta\, \left\langle (\boldsymbol{\nabla}\times\boldsymbol{b})^2\right\rangle/\mu_0$ is the dissipation rate of TME, and

$$\begin{aligned}\Pi_{\rm M} = &\frac{1}{\mu_0}\Big[\langle \boldsymbol{u}\cdot[\boldsymbol{b}\times(\boldsymbol{\nabla}\times\boldsymbol{b})]\rangle - \langle \boldsymbol{u}\times(\boldsymbol{\nabla}\times\boldsymbol{b})\rangle\cdot\overline{\boldsymbol{B}}\\ &+ \langle b_i\, b_j\rangle\, \nabla_j \overline{U}_i - \left\langle \boldsymbol{b}^2\right\rangle \left(\boldsymbol{\nabla}\cdot\overline{\boldsymbol{U}}\right)\Big]\end{aligned} \tag{5}$$

is the production rate of TME. Here η is the magnetic diffusion due to electrical conductivity of the fluid. The first two terms in the RHS of Eq. (5) describe an energy exchange between the turbulent magnetic and kinetic energies. The estimate for the dissipation rate of the turbulent magnetic energy density is $\varepsilon_{\rm M} = E_{\rm M}/\tau_0$.

The density of total turbulent energy (TTE), $E_{\rm T} =$

$E_{\rm K} + E_{\rm M}$, is determined by the following budget equation:

$$\frac{\partial E_{\rm T}}{\partial t} + {\rm div}\, \mathbf{\Phi}_{\rm T} = \Pi_{\rm T} - \varepsilon_{\rm T}, \tag{6}$$

where

$$\Pi_{\rm T} = \biggl[\Bigl(\langle b_i\, b_j\rangle - \mu_0\, \overline{\rho}\, \langle u_i u_j\rangle\Bigr) \nabla_j \overline{U}_i - \langle \boldsymbol{b}^2\rangle\, (\boldsymbol{\nabla}\cdot\overline{\boldsymbol{U}}) - \langle \boldsymbol{u}\times\boldsymbol{b}\rangle \cdot (\boldsymbol{\nabla}\times\overline{\boldsymbol{B}})\biggr]\mu_0^{-1} + \overline{\rho}\,\Bigl(g\, F_z + \langle \boldsymbol{u}\cdot\boldsymbol{f}\rangle\Bigr). \tag{7}$$

is the production rate of $E_{\rm T}$, $\varepsilon_{\rm T} = \varepsilon_{\rm K} + \varepsilon_{\rm M}$ is the dissipation rate of $E_{\rm T}$ and $\mathbf{\Phi}_{\rm T} = \mathbf{\Phi}_{\rm K} + \mathbf{\Phi}_{\rm M}$ is the flux of $E_{\rm T}$.

To determine the production rate of TTE, we use the following second moments for magnetic fluctuations (Rogachevskii & Kleeorin 2007),

$$\langle b_i\, b_j\rangle = \frac{\overline{\boldsymbol{B}}^2}{2}\biggl[2q_{\rm p}\left(\overline{B}\right)\, \delta_{ij} - q_{\rm s}\left(\overline{B}\right)\Bigl(\delta_{ij} + \beta_{ij}\Bigr)\biggr], \tag{8}$$

and velocity fluctuations,

$$\overline{\rho}\,\langle u_i\, u_j\rangle = -\frac{\overline{\boldsymbol{B}}^2}{2\mu_0}\biggl[2q_{\rm p}\left(\overline{B}\right)\, \delta_{ij} - q_{\rm s}\left(\overline{B}\right)\Bigl(\delta_{ij} + \beta_{ij}\Bigr)\biggr] + \overline{\rho}\,\langle u_i\, u_j\rangle^{(0)}, \tag{9}$$

where $\beta_{ij} = \overline{B}_i \overline{B}_j/\overline{\boldsymbol{B}}^2$. The tensor $\langle u_i\, u_j\rangle^{(0)}$ for a background turbulence (with a zero mean magnetic field) in Eq. (9) has two contributions caused by background isotropic velocity fluctuations and tangling anisotropic velocity fluctuations due to the mean velocity shear (Elperin et al. 2002):

$$\langle u_i\, u_j\rangle^{(0)} = \frac{1}{3}\left\langle \boldsymbol{u}^2\right\rangle^{(0)}\, \delta_{ij} - 2\nu_{_T}^{(0)}\, \left(\partial \overline{U}\right)_{ij}, \tag{10}$$

where $\left(\partial \overline{U}\right)_{ij} = (\nabla_i \overline{U}_j + \nabla_j \overline{U}_i)/2$ and $\nu_{_T}^{(0)} = \tau_0 \langle \boldsymbol{u}^2\rangle^{(0)}/3$ is the turbulent viscosity. For simplicity, in Eq. (8) we do not take into account a small-scale dynamo with a zero mean magnetic field.

The nonlinear functions $q_{\rm p}(\overline{B})$ and $q_{\rm s}(\overline{B})$ entering in Eq. (8)–(9) are given by Eqs. (A1)–(A2) in Appendix A. The asymptotic formulae for the nonlinear functions $q_{\rm p}(\overline{B})$ and $q_{\rm s}(\overline{B})$ are as follows. For a very weak mean magnetic field, $\overline{B} \ll \overline{B}_{\rm eq}/4{\rm Rm}^{1/4}$, the nonlinear functions are given by

$$q_{\rm p}(\overline{B}) = \frac{2}{5}\left[\ln {\rm Rm} + \frac{4}{45}\right], \tag{11}$$

$$q_{\rm s}(\overline{B}) = \frac{8}{15}\left[\ln {\rm Rm} + \frac{2}{15}\right], \tag{12}$$

where $\overline{B}_{\rm eq}^2 = \mu_0\, \overline{\rho}\, \langle \boldsymbol{u}^2\rangle$. For $\overline{B}_{\rm eq}/4{\rm Rm}^{1/4} \ll \overline{B} \ll \overline{B}_{\rm eq}/4$, these nonlinear functions are given by

$$q_{\rm p}(\overline{B}) = \frac{16}{25}\left[5|\ln(\sqrt{2}\beta)| + 1 + 4\beta^2\right], \tag{13}$$

$$q_{\rm s}(\overline{B}) = \frac{32}{15}\left[|\ln(\sqrt{2}\beta)| + \frac{1}{30} + \frac{3}{2}\beta^2\right], \tag{14}$$

and for $\overline{B} \gg \overline{B}_{\rm eq}/4$ they are given by

$$q_{\rm p}(\overline{B}) = \frac{4}{3\beta^2}, \quad q_{\rm s}(\overline{B}) = \frac{\pi\sqrt{2}}{3\beta^3}. \tag{15}$$

where $\beta = \sqrt{8}\, \overline{B}/\overline{B}_{\rm eq}$.

Substituting Eqs. (8)–(10) into Eq. (7), we obtain the production rate of TTE as

$$\Pi_{\rm T} = \left[\frac{\overline{\boldsymbol{B}}^2}{2\mu_0}\Bigl(3q_{\rm p}\left(\overline{B}\right) - q_{\rm s}\left(\overline{B}\right)\Bigr) - \frac{\overline{\rho}\,\langle \boldsymbol{u}^2\rangle^{(0)}}{3}\right](\boldsymbol{\nabla}\cdot\overline{\boldsymbol{U}}) + \left[2\nu_{_{\rm T}}\, \overline{\rho}\, \left(\partial \overline{U}\right)_{ij} - \frac{1}{\mu_0}\, q_{\rm s}\left(\overline{B}\right)\, \overline{B}_i \overline{B}_j\right]\, \left(\partial \overline{U}\right)_{ij} - \frac{1}{\mu_0}\, \boldsymbol{\mathcal{E}}\left(\overline{\boldsymbol{B}}\right)\cdot(\boldsymbol{\nabla}\times\overline{\boldsymbol{B}}) + \overline{\rho}\,\Bigl(g\, F_z + \langle \boldsymbol{u}\cdot\boldsymbol{f}\rangle\Bigr), \tag{16}$$

where $\boldsymbol{\mathcal{E}}\left(\overline{\boldsymbol{B}}\right) = \langle \boldsymbol{u}\times\boldsymbol{b}\rangle$ is the turbulent nonlinear electromotive force. Using the steady state solution of Eq. (6), we estimate the total turbulent energy density as $E_{\rm K} + E_{\rm M} \sim \tau_0\, \Pi_{\rm T}$. Equation (8) yields the density of turbulent magnetic energy $E_{\rm M} = \langle \boldsymbol{b}^2\rangle/2\mu_0$ as

$$E_{\rm M} = \left[3q_{\rm p}\left(\overline{B}\right) - 2q_{\rm s}\left(\overline{B}\right)\right]\, \frac{\overline{\boldsymbol{B}}^2}{2\mu_0}. \tag{17}$$

In the next sections, we apply the budget equations for analysis of nonlinear mean-field $\alpha\Omega$, α^2 and $\alpha^2\Omega$ dynamos.

3 MEAN-FIELD $\alpha\Omega$ DYNAMO

In this section, we consider the axisymmetric mean-field $\alpha\Omega$ dynamo, so that the mean magnetic field can be decomposed as

$$\overline{\boldsymbol{B}} = \overline{B}_y(t,x,z)\boldsymbol{e}_y + {\rm rot}[\overline{A}(t,x,z)\boldsymbol{e}_y], \tag{18}$$

and nonlinear mean-field induction equation reads

$$\frac{\partial}{\partial t}\begin{pmatrix}\overline{A}\\ \overline{B}_y\end{pmatrix} = \hat{N}\begin{pmatrix}\overline{A}\\ \overline{B}_y\end{pmatrix}, \tag{19}$$

where the operator $\hat{N}$ is given by

$$\hat{N} = \begin{pmatrix} \eta_{_T}^{(A)}\left(\overline{B}\right)\Delta & \alpha\left(\overline{\boldsymbol{B}}\right) \\ R_\alpha R_\omega\, \hat{\Omega} & \nabla_j\, \eta_{_T}^{(B)}\left(\overline{B}\right)\nabla_j \end{pmatrix}, \tag{20}$$

and the operator

$$\hat{\Omega}\,\overline{A} = \frac{\partial(\delta\Omega\,\sin\vartheta,\, \overline{A})}{\partial(z,\, x)} \tag{21}$$

describes differential rotation. Here ϑ is the angle between $\delta\mathbf{\Omega}$ and the vertical coordinate z and L is the characteristic scale (e.g., the radius of a star or the thickness of a galactic disk). The total α effect is the sum of the kinetic α effect, $\alpha_{\rm K}(\overline{B})$, and the magnetic α effect, $\alpha_{\rm M}(\overline{\boldsymbol{B}})$,

$$\alpha(\overline{\boldsymbol{B}}) = \alpha_{\rm K}(\overline{B}) + \alpha_{\rm M}(\overline{\boldsymbol{B}}), \tag{22}$$

where the kinetic α effect is proportional to the kinetic helicity $H_{\rm u} = \langle \boldsymbol{u}\cdot(\boldsymbol{\nabla}\times\boldsymbol{u})\rangle$, and the magnetic α effect is proportional to the current helicity $H_{\rm c}\left(\overline{\boldsymbol{B}}\right) = \langle \boldsymbol{b}\cdot(\boldsymbol{\nabla}\times\boldsymbol{b})\rangle$ of the small-scale magnetic field $\boldsymbol{b}$ (Pouquet et al. 1976). In particular, the magnetic α effect is given by $\alpha_{\rm M}\left(\overline{\boldsymbol{B}}\right) = \tau_0\, H_{\rm c}\left(\overline{\boldsymbol{B}}\right)\, \phi_{\rm M}\left(\overline{B}\right)/(3\mu_0\,\overline{\rho})$, where $\phi_{\rm M}\left(\overline{B}\right)$ is the algebraic quenching function of the magnetic α effect (Field et al. 1999; Rogachevskii & Kleeorin 2000). The dynamics of the current helicity $H_{\rm c}\left(\overline{B}\right)$ depends on evolution of the small-scale magnetic helicity density $H_{\rm m}\left(\overline{B}\right) = \langle \boldsymbol{a}\cdot\boldsymbol{b}\rangle$, that is determined by a budget equation including the source terms (Kleeorin & Ruzmaikin 1982; Gruzinov & Diamond 1994; Kleeorin et al. 1995) and turbulent fluxes of magnetic helicity (Kleeorin & Rogachevskii

1999; Kleeorin et al. 2000; Blackman & Field 2000; Vishniac & Cho 2001; Brandenburg & Subramanian 2005; Kleeorin & Rogachevskii 2022; Gopalakrishnan & Subramanian 2023). Here $\boldsymbol{b} = \boldsymbol{\nabla}\times\boldsymbol{a}$ are magnetic fluctuations and $\boldsymbol{a}$ are fluctuations of magnetic vector potential.

Taking into account turbulent fluxes of the small-scale magnetic helicity, it has been shown by numerical simulations that a nonlinear galactic dynamo governed by a dynamic equation for the magnetic helicity density $H_{\rm m}$ of a small-scale field (the dynamical nonlinearity) saturates at a mean magnetic field comparable with the equipartition magnetic field (see, e.g., Kleeorin et al. 2000, 2002, 2003b,a; Blackman & Brandenburg 2002; Brandenburg & Subramanian 2005; Shukurov et al. 2006). Numerical simulations demonstrate that the dynamics of magnetic helicity plays a crucial role in solar dynamo as well (see, e.g., Kleeorin et al. 2003b, 2016, 2020; Sokoloff et al. 2006; Zhang et al. 2006, 2012; Käpylä et al. 2010; Hubbard & Brandenburg 2012; Del Sordo et al. 2013; Safiullin et al. 2018; Rincon 2021). Different forms of magnetic helicity fluxes have been suggested in various studies using phenomenological arguments (Kleeorin & Rogachevskii 1999; Kleeorin et al. 2000, 2002; Vishniac & Cho 2001; Subramanian & Brandenburg 2004; Brandenburg & Subramanian 2005). Recently, the turbulent magnetic helicity fluxes have been rigorously derived (Kleeorin & Rogachevskii 2022; Gopalakrishnan & Subramanian 2023). In particular, Kleeorin & Rogachevskii (2022) apply the mean-field theory, adopt the Coulomb gauge and consider a strongly density-stratified turbulence. They have found that the turbulent magnetic helicity fluxes depend on the mean magnetic field energy, and include non-gradient and gradient contributions. In addition, Gopalakrishnan & Subramanian (2023) have recently shown that contributions to the turbulent magnetic helicity fluxes from the third-order moments can be described using the turbulent diffusion approximation.

The kinetic α effect is given by $\alpha_{_{\rm K}}\left(\overline{B}\right) = \alpha_{_{\rm K}}^{(0)}\,\phi_{_{\rm K}}\left(\overline{B}\right)$ (Rogachevskii & Kleeorin 2004), where for a forced turbulence $\alpha_{_{\rm K}}^{(0)} = -\tau_0\, H_{\rm u}/3$ and the algebraic quenching function $\phi_{_{\rm K}}\left(\overline{B}\right)$ of the kinetic α effect has the following asymptotic behavior: $\phi_{_{\rm K}} = 1$ when $\overline{B} \ll \overline{B}_{\rm eq}/4$ and $\phi_{_{\rm K}} = (1/4)\,(\overline{B}/\overline{B}_{\rm eq})^{-2}$ when $\overline{B} \gg \overline{B}_{\rm eq}/4$. The similar asymptotic behavior is also for the algebraic quenching of the magnetic α effect.

The turbulent magnetic diffusion of the toroidal mean magnetic field is given by (Rogachevskii & Kleeorin 2004): $\eta_{_{T}}^{(B)}\left(\overline{B}\right) = \eta_{_{T}}^{(0)}\,\phi_\eta^{(B)}\left(\overline{B}\right)$, where $\eta_{_{T}}^{(0)} = \tau_0 \langle\boldsymbol{u}^2\rangle^{(0)}/3$, the algebraic quenching function $\phi_\eta^{(B)}\left(\overline{B}\right)$ of the toroidal mean magnetic field is $\phi_\eta^{(B)} = 1$ when $\overline{B} \ll \overline{B}_{\rm eq}/4$ and $\phi_\eta^{(B)} = (1/4)\,(\overline{B}/\overline{B}_{\rm eq})^{-1}$ when $\overline{B} \gg \overline{B}_{\rm eq}/4$. The similar asymptotic behavior is also for the turbulent viscosity (Rogachevskii & Kleeorin 2004). The turbulent magnetic diffusion of the poloidal mean magnetic field behaves as (Rogachevskii & Kleeorin 2004): $\eta_{_{T}}^{(A)}\left(\overline{B}\right) = \eta_{_{T}}^{(0)}\,\phi_\eta^{(A)}\left(\overline{B}\right)$, where the algebraic quenching function $\phi_\eta^{(A)}\left(\overline{B}\right)$ of the poloidal mean magnetic field is $\phi_\eta^{(A)} = 1$ when $\overline{B} \ll \overline{B}_{\rm eq}/4$ and $\phi_\eta^{(A)} = (1/8)\,(\overline{B}/\overline{B}_{\rm eq})^{-2}$ when $\overline{B} \gg \overline{B}_{\rm eq}/4$.

Equations (19)–(21) are written in dimensionless variables: the coordinate is measured in the units of L, the time t is is measured in the units of turbulent magnetic diffusion time $L^2/\eta_{_{T}}^{(0)}$; the mean magnetic field is measured in the units of $\overline{B}_*$, where $\overline{B}_* \equiv \sigma\,\overline{B}_*^{\rm eq}$, $\sigma = \ell_0/\sqrt{2}L$, $\overline{B}_*^{\rm eq} = u_0\,\sqrt{\mu_0\overline{\rho}_*}$, and the magnetic potential, $\overline{A}$ is measured in the units of $R_\alpha L\overline{B}_*$. Here $R_\alpha = \alpha_* L/\eta_{_{T}}^{(0)}$, the fluid density $\overline{\rho}$ is measured in the units $\overline{\rho}_*$, the differential rotation $\delta\Omega$ is measured in units of the maximal value of the angular velocity Ω, the α effect is measured in units of the maximum value of the kinetic α effect, α_*; the integral scale of the turbulent motions $\ell_0 = \tau_0\, u_0$ and the characteristic turbulent velocity $u_0 = \sqrt{\langle\boldsymbol{u}^2\rangle^{(0)}}$ at the scale ℓ_0 are measured in units of their maximum values in the turbulent region, and the turbulent magnetic diffusion coefficients are measured in units of their maximum values. The magnetic Reynolds number ${\rm Rm} = \ell_0\, u_0/\eta$ is defined using the maximal values of the integral scale ℓ_0 and the characteristic turbulent velocity u_0. The dynamo number for the linear $\alpha\Omega$ dynamo is defined as $D_{\rm L} = R_\alpha R_\omega$, where $R_\omega = (\delta\Omega)\, L^2/\eta_{_{T}}^{(0)}$.

Now we define the nonlinear dynamo number $D_{\rm N}\left(\overline{B}\right)$ for the $\alpha\Omega$ dynamo as

$$
D_{\rm N}\left(\overline{B}\right) = \frac{\alpha\left(\overline{B}\right)\,\delta\Omega\, L^3}{\eta_{_{T}}^{(B)}\left(\overline{B}\right)\,\eta_{_{T}}^{(A)}\left(\overline{B}\right)}, \tag{23}
$$

where we take into account that the nonlinear turbulent magnetic diffusion coefficients of the poloidal and toroidal components of the mean magnetic field are different (Rogachevskii & Kleeorin 2004). The ratio of energies of the toroidal and poloidal mean magnetic fields for the $\alpha\Omega$ dynamos is of the order of $D_{\rm L}^2/D_{\rm cr}$, where $D_{\rm cr}$ is the threshold for the excitation of the $\alpha\Omega$ dynamo.

Next, we take into account the feedback of the mean magnetic field on the background turbulence using the budget equation for the total turbulent energy. In a shear-produced non-convective turbulence, the largest contributions to the production rate of TTE for a strong large-scale magnetic field is due to the terms $-\boldsymbol{\mathcal{E}}\left(\overline{\boldsymbol{B}}\right)\cdot(\boldsymbol{\nabla}\times\overline{\boldsymbol{B}})/\mu_0$ and $2\nu_{_{T}}\left(\overline{B}\right)\,\overline{\rho}\,\left(\partial\overline{U}\right)_{ij}^2 \equiv 2\nu_{_{T}}\,\overline{\rho}\, S^2$ [see Eq. (16)], where $S^2 = \left(\partial\overline{U}\right)_{ij}^2$. This implies that the turbulent kinetic energy density for a strong large-scale magnetic field is estimated as

$$
E_{\rm K} = \tau_0\left[2\nu_{_{T}}\left(\overline{B}\right)\,\overline{\rho}\, S^2 - \frac{1}{\mu_0}\,\boldsymbol{\mathcal{E}}\left(\overline{\boldsymbol{B}}\right)\cdot(\boldsymbol{\nabla}\times\overline{\boldsymbol{B}})\right]. \tag{24}
$$

Therefore, the turbulent kinetic energy density for strong mean magnetic fields behaves as

$$
E_{\rm K}\left(\overline{B}\right) \approx E_{\rm K}^{(0)}\left[1 + \frac{D_{\rm cr}^{1/2}}{D_{\rm L}}\left(\frac{\ell_0}{L_B}\right)^2\left(\frac{\overline{B}}{\overline{B}_{\rm eq}}\right)^2\right], \tag{25}
$$

where $E_{\rm K}^{(0)} = (2/3)\,\overline{\rho}\,\ell_0^2\, S^2$ and the characteristic scale of the mean magnetic field variations L_B is defined as $L_B = \overline{B}/|\boldsymbol{\nabla}\times\overline{\boldsymbol{B}}|$. We also take into account that for strong mean magnetic fields, the ratio of these production terms is

$$
-\frac{\tau_0}{E_{\rm K}^{(0)}}\boldsymbol{\mathcal{E}}\left(\overline{\boldsymbol{B}}\right)(\boldsymbol{\nabla}\times\overline{\boldsymbol{B}}) \propto \frac{D_{\rm cr}^{1/2}}{D_{\rm L}}\left(\frac{\ell_0}{L_B}\right)^2\left(\frac{\overline{B}}{\overline{B}_{\rm eq}}\right)^2. \tag{26}
$$

This yields the estimate for the ratio $\eta_{_{T}}^{(B)}\left(\overline{B}\right)/\eta_{_{T}}^{(0)}$ for strong mean magnetic fields as

$$\frac{\eta_{_T}^{(B)}\left(\overline{B}\right)}{\eta_{_T}^{(0)}} \approx \frac{1}{4}\left[1+\frac{D_{\rm cr}^{1/2}}{D_{\rm L}}\left(\frac{\ell_0}{L_B}\right)^2\left(\frac{\overline{B}}{\overline{B}_{\rm eq}}\right)^2\right]\left(\frac{\overline{B}}{\overline{B}_{\rm eq}}\right)^{-1}, \tag{27}$$

where the ratio of turbulent diffusion coefficients of poloidal and toroidal fields $\eta_{_T}^{(A)}\left(\overline{B}\right)/\eta_{_T}^{(B)}\left(\overline{B}\right)$ is given by

$$\frac{\eta_{_T}^{(A)}\left(\overline{B}\right)}{\eta_{_T}^{(B)}\left(\overline{B}\right)} \approx \frac{1}{2}\left(\frac{\overline{B}}{\overline{B}_{\rm eq}}\right)^{-1}, \tag{28}$$

and $\eta_{_T}^{(A,B)}\left(\overline{B}\right) = 2\tau_0\, E_{\rm K}\left(\overline{B}\right)\phi_\eta^{(A,B)}/3\overline{\rho}$. Therefore, the ratio of the nonlinear and linear dynamo numbers $D_{\rm N}\left(\overline{B}\right)/D_{\rm L}$ in a shear-produced non-convective turbulence for strong mean magnetic fields is estimated as

$$\frac{D_{\rm N}\left(\overline{B}\right)}{D_{\rm L}} \approx 32\left[1+\frac{D_{\rm cr}^{1/2}}{D_{\rm L}}\left(\frac{\ell_0}{L_B}\right)^2\left(\frac{\overline{B}}{\overline{B}_{\rm eq}}\right)^2\right]^{-2} \times\frac{\alpha\left(\overline{B}\right)}{\alpha_{_{\rm K}}^{(0)}}\left(\frac{\overline{B}}{\overline{B}_{\rm eq}}\right)^3, \tag{29}$$

where the dependence of the total α effect on the mean magnetic field, $\alpha\left(\overline{B}\right)$, is caused by the algebraic and dynamic quenching.

The algebraic quenching describes the feedback of the mean magnetic field on the plasma motions, while the dynamic quenching of the total α effect is caused by the evolution of the magnetic α effect related to the small-scale current and magnetic helicities. In particular, the dynamic equation for the small-scale current helicity (which determines the evolution of the magnetic α effect) in a steady state yields the total α effect as $\alpha\left(\overline{B}\right) \propto -{\rm div}\boldsymbol{F}_{\rm M}/\overline{B}^2$, where $\boldsymbol{F}_{\rm M}$ is the magnetic helicity flux of the small-scale magnetic field. This implies that if $\boldsymbol{F}_{\rm M}$ does not quenched with the growth of the mean magnetic field, the total α effect for strong magnetic fields behaves as $\alpha\left(\overline{B}\right) \propto \left(\overline{B}/\overline{B}_{\rm eq}\right)^{-2}$. In the case of the algebraic quenching of the magnetic helicity flux $\boldsymbol{F}_{\rm M}$, the dependence of $\alpha\left(\overline{B}\right)$ with the growth of the mean magnetic field is more stronger, i.e., $\alpha\left(\overline{B}\right)/\alpha_{_{\rm K}}^{(0)} \propto \left(\overline{B}/\overline{B}_{\rm eq}\right)^{-n}$ with $n>2$. Equation (29) implies that the nonlinear dynamo number decreases for any strong values of the mean magnetic field for a shear-produced non-convective turbulence, resulting in saturation of the mean-field dynamo instability.

In a convective turbulence, the largest contributions to the production rate of TTE for a strong mean magnetic fields is due to the buoyancy term $\overline{\rho}\, g\, F_z$ and the term $\eta_{_T}^{(B)}\left(\overline{B}\right)(\boldsymbol{\nabla}\times\overline{\boldsymbol{B}})^2/\mu_0$ [see Eq. (16)]. This implies that the turbulent kinetic energy density is given by

$$E_{\rm K} = \tau_0\left[\overline{\rho}\, g\, F_z - \frac{1}{\mu_0}\boldsymbol{\mathcal{E}}\left(\overline{\boldsymbol{B}}\right)\cdot(\boldsymbol{\nabla}\times\overline{\boldsymbol{B}})\right], \tag{30}$$

where $\tau_0 = \ell_0\left[2E_{\rm K}/\overline{\rho}\right]^{-1/2}$. Thus, Eq. (30) can be rewritten as the following nonlinear equation:

$$\tilde{E}_{\rm K}^{3/2} - \xi\left(\overline{B}\right)\tilde{E}_{\rm K}^{1/2} - 1 = 0, \tag{31}$$

where $\tilde{E}_{\rm K} = E_{\rm K}/E_{\rm K}^{(0)}$,

$$E_{\rm K}^{(0)} = \frac{\overline{\rho}}{2}\left(2g\, F_z\, \ell_0\right)^{2/3}, \tag{32}$$

$$\xi\left(\overline{B}\right) = \frac{2}{3}\frac{D_{\rm cr}^{1/2}}{D_{\rm L}}\left(\frac{\ell_0}{L_B}\right)^2\left(\frac{\overline{B}}{\overline{B}_{\rm eq}}\right)^2, \tag{33}$$

and $\overline{B}_{\rm eq}^2 = 2\mu_0 E_{\rm K}^{(0)}$. Nonlinear equation (31) has the following asymptotic solution: $\tilde{E}_{\rm K}^{1/2} = 1$ for $\xi\left(\overline{B}\right)\tilde{E}_{\rm K}^{1/2} \ll 1$, and $\tilde{E}_{\rm K}^{1/2} = \xi\left(\overline{B}\right)$ for $\xi\left(\overline{B}\right)\tilde{E}_{\rm K}^{1/2} \gg 1$. Thus, an approximate solution of the nonlinear equation (31) can be constructed as a linear combination of these asymptotic solutions, i.e., the turbulent kinetic energy density for strong mean magnetic fields behaves as

$$E_{\rm K} \approx E_{\rm K}^{(0)}\left[1+\frac{D_{\rm cr}^{1/2}}{D_{\rm L}}\left(\frac{\ell_0}{L_B}\right)^2\left(\frac{\overline{B}}{\overline{B}_{\rm eq}}\right)^2\right]. \tag{34}$$

This implies that equation (34) for the turbulent kinetic energy density for strong mean magnetic fields for convective turbulence is similar to Eq. (25) derived for a shear-produced non-convective turbulence. The difference is only in equation for $E_{\rm K}^{(0)}$ that is given by Eq. (32) for a convective turbulence. Therefore, equations for the ratios $\eta_{_T}^{(B)}\left(\overline{B}\right)/\eta_{_T}^{(0)}$, $\eta_{_T}^{(A)}\left(\overline{B}\right)/\eta_{_T}^{(B)}\left(\overline{B}\right)$ and $D_{\rm N}\left(\overline{B}\right)/D_{\rm L}$ in a convective turbulence for strong mean magnetic fields are similar to Eqs. (27)–(29), respectively.

In a forced turbulence, the turbulent kinetic energy density for a strong mean magnetic field is given by

$$E_{\rm K} = \tau_0\left[\overline{\rho}\,\langle\boldsymbol{u}\cdot\boldsymbol{f}\rangle - \frac{1}{\mu_0}\boldsymbol{\mathcal{E}}\left(\overline{\boldsymbol{B}}\right)\cdot(\boldsymbol{\nabla}\times\overline{\boldsymbol{B}})\right], \tag{35}$$

where we take into account that the largest contribution to the production rate of TTE in a non-convective forced turbulence for a strong mean magnetic field is due to the terms $-\boldsymbol{\mathcal{E}}\left(\overline{\boldsymbol{B}}\right)\cdot(\boldsymbol{\nabla}\times\overline{\boldsymbol{B}})/\mu_0$ and $\overline{\rho}\,\langle\boldsymbol{u}\cdot\boldsymbol{f}\rangle$ [see Eq. (16)]. Therefore, the turbulent kinetic energy density for strong mean magnetic fields behaves as

$$E_{\rm K} \approx E_{\rm K}^{(0)}\left[1+\frac{1}{8}\frac{D_{\rm cr}^{1/2}}{D_{\rm L}}\left(\frac{\ell_0}{L_B}\right)^2\left(\frac{\overline{B}}{\overline{B}_{\rm eq}}\right)\right], \tag{36}$$

where $E_{\rm K}^{(0)} = \overline{\rho}\,\tau_0\,\langle\boldsymbol{u}\cdot\boldsymbol{f}\rangle$. This yields the estimates for the ratio $\eta_{_T}^{(B)}\left(\overline{B}\right)/\eta_{_T}^{(0)}$ as

$$\frac{\eta_{_T}^{(B)}\left(\overline{B}\right)}{\eta_{_T}^{(0)}} \approx \frac{1}{4}\left[1+\frac{D_{\rm cr}^{1/2}}{8D_{\rm L}}\left(\frac{\ell_0}{L_B}\right)^2\left(\frac{\overline{B}}{\overline{B}_{\rm eq}}\right)\right]\left(\frac{\overline{B}}{\overline{B}_{\rm eq}}\right)^{-1}, \tag{37}$$

where the ratio $\eta_{_T}^{(A)}\left(\overline{B}\right)/\eta_{_T}^{(B)}\left(\overline{B}\right)$ is given by Eq. (28). Using Eq. (37), we determine the ratio of nonlinear and linear dynamo numbers $D_{\rm N}\left(\overline{B}\right)/D_{\rm L}$ in a non-convective forced turbulence for strong mean magnetic fields as

$$\frac{D_{\rm N}\left(\overline{B}\right)}{D_{\rm L}} \approx 32\left[1+\frac{1}{8}\frac{D_{\rm cr}^{1/2}}{D_{\rm L}}\left(\frac{\ell_0}{L_B}\right)^2\left(\frac{\overline{B}}{\overline{B}_{\rm eq}}\right)\right]^{-2} \times\frac{\alpha\left(\overline{B}\right)}{\alpha_{_{\rm K}}^{(0)}}\left(\frac{\overline{B}}{\overline{B}_{\rm eq}}\right)^3. \tag{38}$$

Equations (29) and (38) imply that for the $\alpha\Omega$ dynamo, the nonlinear dynamo number decreases with increase of the mean magnetic field for a forced turbulence, and a shear-produced turbulence and a convective turbulence. This causes saturation of the mean-field $\alpha\Omega$ dynamo instability for a strong mean magnetic field.

4 MEAN-FIELD α^2 DYNAMO

In this section, we consider mean-field α^2 dynamo. First, we discuss a long-standing question when a one-dimensional kinematic α^2 dynamo can be oscillatory. The mean magnetic field $\overline{\boldsymbol{B}}(t,z) = \boldsymbol{\nabla} \times \overline{\boldsymbol{A}} = (-\nabla_z \overline{A}_y, \nabla_z \overline{A}_x, 0)$ is determined by the following equation

$$
\frac{\partial \Psi}{\partial t} = \hat{L}\Psi, \tag{39}
$$

where $\overline{\boldsymbol{A}}$ is the mean magnetic vector potential in the Weyl gauge. The linear operator $\hat{L}$ and the function $\Psi(t,z)$ are given by

$$
\hat{L} \quad = \quad \begin{pmatrix} \eta_T^{(0)} \nabla_z^2 & -\alpha_{\rm K}^{(0)} \nabla_z \\ \alpha_{\rm K}^{(0)} \nabla_z & \eta_T^{(0)} \nabla_z^2 \end{pmatrix}, \quad \Psi = \begin{pmatrix} A_x \\ A_y \end{pmatrix}, \tag{40}
$$

where $\eta_T^{(0)}$ is the turbulent magnetic diffusion coefficient, and $\alpha_{\rm K}^{(0)}$ is the kinetic α effect caused by the helical turbulent motions in plasma.

When can a one-dimensional kinematic α^2 dynamo be oscillatory? First, if the linear operator $\hat{L}$ is not self-adjoint, it has complex eigenvalues. This case corresponds to the oscillatory growing solution, i.e., the dynamo is oscillatory. On the other hand, any self-adjoint operator, $\hat{M}$, defining by the following condition,

$$
\int \Psi^* \hat{M} \tilde{\Psi} \, dz = \int \tilde{\Psi} \hat{M}^* \Psi^* \, dz, \tag{41}
$$

has real eigenvalues, where the asterisk denotes complex conjugation. Now we determine conditions when the linear operator $\hat{L}$ is not self-adjoint, i.e., it has complex eigenvalues. To this end, we determine the integrals $\int \Psi^* \hat{L} \tilde{\Psi} \, dz$ and $\int \tilde{\Psi} \hat{L}^* \Psi^* \, dz$ as:

$$
\begin{aligned}
\int \Psi^* \hat{L} \tilde{\Psi} \, dz &= \int \alpha_{\rm K}^{(0)} \left(A_y^* \nabla_z \tilde{A}_x - A_x^* \nabla_z \tilde{A}_y \right) dz \\
&- \int \eta_T^{(0)} \left[\left(\nabla_z A_x^*\right) \nabla_z \tilde{A}_x + \left(\nabla_z A_y^*\right) \nabla_z \tilde{A}_y \right] dz \\
&+ \left[\eta_T^{(0)} \left(A_x^* \, \nabla_z \tilde{A}_x + A_y^* \, \nabla_z \tilde{A}_y \right) \right]_{z=L_{\rm bott}}^{z=L_{\rm top}},
\end{aligned} \tag{42}
$$

$$
\begin{aligned}
\int \tilde{\Psi} \hat{L}^* \Psi^* \, dz &= \int \alpha_{\rm K}^{(0)} \left(A_y^* \nabla_z \tilde{A}_x - A_x^* \nabla_z \tilde{A}_y \right) dz \\
&- \int \eta_T^{(0)} \left[\left(\nabla_z A_x^*\right) \nabla_z \tilde{A}_x + \left(\nabla_z A_y^*\right) \nabla_z \tilde{A}_y \right] dz \\
&+ \Bigg[\eta_T^{(0)} \left(\tilde{A}_x \, \nabla_z A_x^* + \tilde{A}_y \, \nabla_z A_y^* \right) + \alpha_{\rm k} \Big(A_x^* \tilde{A}_y \\
&- A_y^* \, \tilde{A}_x \Big) \Bigg]_{z=L_{\rm bott}}^{z=L_{\rm top}},
\end{aligned} \tag{43}
$$

where $z = L_{\rm bott}$ and $z = L_{\rm top}$ are the bottom and upper boundaries, respectively. When $\eta_T^{(0)}$ and $\alpha_{\rm K}^{(0)}$ vanish at the boundaries where the turbulence is very weak, the operator $\hat{L}$ satisfies condition (41) and the α^2 dynamo is not oscillatory. On the other hand, when $\alpha_{\rm K}^{(0)}$ vanishes only at one boundary, while it is non-zero at the other boundary, the operator $\hat{L}$ does not satisfy condition (41), and the α^2 dynamo is oscillatory. The latter case has been considered in analytical study by Shukurov et al. (1985); Rädler & Bräuer (1987) and in numerical study by Baryshnikova & Shukurov (1987). Brandenburg (2017) has recently considered the one-dimensional kinematic α^2 dynamo with different conditions at two boundaries: $\boldsymbol{A} = 0$ at $z = L_{\rm bott}$ and $\nabla_z \boldsymbol{A} = 0$ at $z = L_{\rm top}$, so that the operator $\hat{L}$ may not satisfy condition (41), and the α^2 dynamo may be oscillatory.

Now we consider the nonlinear axisymmetric mean-field α^2 dynamo, so that nonlinear mean-field induction equation reads

$$
\frac{\partial}{\partial t} \begin{pmatrix} \overline{A} \\ \overline{B}_y \end{pmatrix} = \hat{N} \begin{pmatrix} \overline{A} \\ \overline{B}_y \end{pmatrix}, \tag{44}
$$

where the mean magnetic field is $\overline{\boldsymbol{B}} = \overline{B}_y(t,x,z)\boldsymbol{e}_y + {\rm rot}[\overline{A}(t,x,z)\boldsymbol{e}_y]$, the operator $\hat{N}$ is given by

$$
\hat{N} \quad = \quad \begin{pmatrix} \eta_T^{(A)} \left(\overline{B}\right) \Delta & \alpha\left(\overline{B}\right) \\ -R_\alpha^2 \nabla_j \alpha \left(\overline{B}\right) \nabla_j & \nabla_j \eta_T^{(B)} \left(\overline{B}\right) \nabla_j \end{pmatrix}, \tag{45}
$$

and the total α effect is given by $\alpha\left(\overline{B}\right) = \alpha_{\rm K}\left(\overline{B}\right) + \alpha_{\rm M}\left(\overline{\boldsymbol{B}}\right)$. Now we introduce the effective dynamo number $D_{\rm N}^{(\alpha)}\left(\overline{B}\right)$ in the nonlinear α^2 dynamo defined as $D_{\rm N}^{(\alpha)}\left(\overline{B}\right) = \alpha^2\left(\overline{B}\right) L^2 / [\eta_T^{(B)}\left(\overline{B}\right) \eta_T^{(A)}\left(\overline{B}\right)]$. Similarly, the effective dynamo number for a linear α^2 dynamo is defined as $D_{\rm L}^{(\alpha)} = R_\alpha^2$, where $R_\alpha = \alpha_* L / \eta_T^{(0)}$, α_* is the maximum value of the kinetic α effect and L is the stellar radius or the thickness of the galactic disk.

Since poloidal and toroidal components of the mean magnetic field in the nonlinear α^2 mean-field dynamo are of the same order of magnitude, Eqs. (29) and (38) obtained in Section 3 for $\alpha\Omega$ mean-field dynamo can be used for the nonlinear α^2 mean-field dynamo except for they should not contain the ratio $D_{\rm cr}^{1/2}/D_{\rm L}$ (which is the ratio of energies of the poloidal and toroidal mean magnetic fields). Therefore, in a shear-produced non-convective turbulence and in a convective turbulence, the ratio $D_{\rm N}^{(\alpha)}\left(\overline{B}\right)/D_{\rm L}^{(\alpha)}$ for strong mean magnetic fields is given by

$$
\begin{aligned}
\frac{D_{\rm N}^{(\alpha)}}{D_{\rm L}^{(\alpha)}} \quad &\approx \quad 32 \left[1 + \left(\frac{\ell_0}{L_B}\right)^2 \left(\frac{\overline{B}}{\overline{B}_{\rm eq}}\right)^2 \right]^{-2} \\
&\times \left(\frac{\alpha\left(\overline{B}\right)}{\alpha_{\rm K}^{(0)}} \right)^2 \left(\frac{\overline{B}}{\overline{B}_{\rm eq}}\right)^3,
\end{aligned} \tag{46}
$$

while for forced turbulence, the ratio $D_{\rm N}^{(\alpha)}\left(\overline{B}\right)/D_{\rm L}^{(\alpha)}$ for strong mean magnetic fields is given by

$$
\begin{aligned}
\frac{D_{\rm N}^{(\alpha)}\left(\overline{B}\right)}{D_{\rm L}^{(\alpha)}} \quad &\approx \quad 32 \left[1 + \frac{1}{8} \left(\frac{\ell_0}{L_B}\right)^2 \left(\frac{\overline{B}}{\overline{B}_{\rm eq}}\right) \right]^{-2} \\
&\times \left(\frac{\alpha\left(\overline{B}\right)}{\alpha_{\rm K}^{(0)}} \right)^2 \left(\frac{\overline{B}}{\overline{B}_{\rm eq}}\right)^3.
\end{aligned} \tag{47}
$$

These equations take into account the feedback of the mean magnetic field on the background turbulence by means of the budget equation for the total turbulent energy. Thus, Eqs. (46)–(47) imply that for the α^2 dynamo, the nonlinear dynamo number decreases with increase of the mean magnetic field for a forced turbulence, and a shear-produced turbulence and a convective turbulence. This causes a saturation of the mean-field α^2 dynamo instability for a strong mean magnetic field.

5 MEAN-FIELD $\alpha^2\Omega$ DYNAMO

In this section, we consider the axisymmetric mean-field $\alpha^2\Omega$ dynamo, so that and nonlinear mean-field induction equation reads

$$\frac{\partial}{\partial t}\left(\frac{\overline{A}}{\overline{B}_y}\right) = \hat{N}\left(\frac{\overline{A}}{\overline{B}_y}\right), \tag{48}$$

where the mean magnetic field is $\overline{\boldsymbol{B}} = \overline{B}_y(t,x,z)\boldsymbol{e}_y + \mathrm{rot}[\overline{A}(t,x,z)\boldsymbol{e}_y]$, the operator $\hat{N}$ is

$$\hat{N} = \begin{pmatrix} \eta_{_T}^{(A)}\left(\overline{B}\right)\Delta & \alpha\left(\overline{\boldsymbol{B}}\right) \\ R_\alpha\left[R_\omega\hat{\Omega} - R_\alpha\nabla_j\alpha\left(\overline{B}\right)\nabla_j\right] & \nabla_j\eta_{_T}^{(B)}\left(\overline{B}\right)\nabla_j \end{pmatrix}, \tag{49}$$

$R_\alpha = \alpha_* L/\eta_{_T}^{(0)}$ and $R_\omega = (\delta\Omega)\,L^2/\eta_{_T}^{(0)}$.

First, we consider a kinematic dynamo problem, assuming for simplicity that the kinetic α effect is a constant, and the mean velocity $\overline{\boldsymbol{U}} = (0, Sz, 0)$. We seek a solution for Eq. (48) as a real part of the following functions:

$$\overline{A} = A_0\exp[\tilde{\gamma}t - \mathrm{i}\,(k_x x + k_z z))], \tag{50}$$

$$\overline{B}_\varphi = B_0\exp[\tilde{\gamma}t - \mathrm{i}\,(k_x x + k_z z))], \tag{51}$$

where $\tilde{\gamma} = \gamma + \mathrm{i}\,\omega$. Equations (48)–(51) yield the growth rate of the dynamo instability and the frequency of the dynamo waves as

$$\gamma = \frac{R_\alpha R_\alpha^{\rm cr}}{\sqrt{2}}\left[\left[1 + \left(\frac{\zeta R_\omega}{R_\alpha R_\alpha^{\rm cr}}\right)^2\right]^{1/2} + 1\right]^{1/2} - \left(R_\alpha^{\rm cr}\right)^2, \tag{52}$$

$$\omega = -\mathrm{sgn}(R_\omega)\,\frac{R_\alpha R_\alpha^{\rm cr}}{\sqrt{2}}\left[\left[1 + \left(\frac{\zeta R_\omega}{R_\alpha R_\alpha^{\rm cr}}\right)^2\right]^{1/2} - 1\right]^{1/2}, \tag{53}$$

where $\zeta^2 = 1 - (k_x/R_\alpha^{\rm cr})^2$. Here we took into account that $(x+\mathrm{i}y)^{1/2} = \pm(X+\mathrm{i}Y)$, where $X = 2^{-1/2}\,[(x^2+y^2)^{1/2}+x]^{1/2}$ and $Y = \mathrm{sgn}(y)\,2^{-1/2}\,[(x^2+y^2)^{1/2}-x]^{1/2}$. Here the threshold $R_\alpha^{\rm cr}$ for the mean-field dynamo instability, defined by the conditions $\gamma = 0$ and $R_\omega = 0$, is given by $R_\alpha^{\rm cr} = (k_x^2 + k_z^2)^{1/2}$.

Equations (48)–(51) also yield the squared ratio of amplitudes $|A_0/B_0|^2$,

$$\left|\frac{A_0}{B_0}\right|^2 = \left(R_\alpha R_\alpha^{\rm cr}\right)^{-2}\,\left(1 + \zeta^2 R_\omega^2\right)^{-1}, \tag{54}$$

and the phase shift between the toroidal $\overline{B}_\varphi$ and poloidal $\overline{B}_{\rm pol}$ components of the mean magnetic field,

$$\sin(2\delta) = -\zeta R_\omega\,\left[\left(R_\alpha R_\alpha^{\rm cr}\right)^2 + \zeta^2 R_\omega^2\right]^{-1/2}, \tag{55}$$

where $\overline{B}_{\rm pol} = R_\alpha R_\alpha^{\rm cr}\,\overline{A}$. Equation (54) yields the energy ratio of poloidal $\overline{B}_{\rm pol}$ and toroidal $\overline{B}_\varphi$ mean magnetic field components as

$$\frac{\overline{B}_{\rm pol}^2}{\overline{B}_\varphi^2} = \left(1 + \zeta^2 R_\omega^2\right)^{-1}. \tag{56}$$

Asymptotic formulas for the growth rate of the dynamo instability and the frequency of the dynamo waves for a weak differential rotation, $\zeta R_\omega \ll R_\alpha R_\alpha^{\rm cr}$, are given by

$$\gamma = R_\alpha R_\alpha^{\rm cr}\left[1 + \frac{1}{8}\left(\frac{\zeta R_\omega}{R_\alpha R_\alpha^{\rm cr}}\right)^2\right] - \left(R_\alpha^{\rm cr}\right)^2, \tag{57}$$

$$\omega = -\frac{\zeta R_\omega}{\sqrt{2}}. \tag{58}$$

In this case, the mean-field α^2 dynamo is slightly modified by a weak differential rotation, and the phase shift between the fields $\overline{B}_\varphi$ and $\overline{B}_{\rm pol}$ vanishes, while $\overline{B}_{\rm pol}/\overline{B}_\varphi \sim 1$ [see Eqs. (55)–(56)]. In the opposite case, for a strong differential rotation, $\zeta R_\omega \gg R_\alpha R_\alpha^{\rm cr}$, the growth rate of the dynamo instability and the frequency of the dynamo waves are given by

$$\gamma = \left[\frac{1}{2}\,\zeta\,R_\alpha^{\rm cr}\,R_\alpha|R_\omega|\right]^{1/2} - \left(R_\alpha^{\rm cr}\right)^2, \tag{59}$$

$$\omega = -\mathrm{sgn}(R_\omega)\,\left[\frac{1}{2}\,\zeta\,R_\alpha^{\rm cr}\,R_\alpha|R_\omega|\right]^{1/2}. \tag{60}$$

In this case, the mean-field $\alpha\Omega$ dynamo is slightly modified by a weak α^2 effect, and the phase shift between the fields $\overline{B}_\varphi$ and $\overline{B}_{\rm pol}$ tends to $-\pi/4$, while $\overline{B}_{\rm pol}/\overline{B}_\varphi \ll 1$ [see Eqs. (55)–(56)].

The necessary condition for the dynamo ($\gamma > 0$) reads:

- when $R_\alpha/R_\alpha^{\rm cr} < \sqrt{2}$, the mean-field $\alpha^2\,\Omega$ dynamo is excited when

$$D_{\rm L} > \frac{2}{\zeta}\,\left(R_\alpha^{\rm cr}\right)^3; \tag{61}$$

- when $R_\alpha/R_\alpha^{\rm cr} > \sqrt{2}$, the mean-field $\alpha^2\,\Omega$ dynamo is excited for any differential rotation, R_ω. Here $D_{\rm L} = R_\alpha\,R_\omega$.

Analysis which is similar to that performed in Section 3 yields the ratio of the nonlinear and linear dynamo numbers $D_{\rm N}\left(\overline{B}\right)/D_{\rm L}$ in the nonlinear $\alpha^2\Omega$ dynamo for strong mean magnetic fields in a shear-produced and a convective turbulence as

$$\frac{D_{\rm N}\left(\overline{B}\right)}{D_{\rm L}} \approx 32\left[1 + \frac{D_{\rm cr}^{1/2}}{D_{\rm L}}\left(\frac{\ell_0}{L_B}\right)^2\left(\frac{\overline{B}}{\overline{B}_{\rm eq}}\right)^2\right]^{-2} \times \frac{\alpha\left(\overline{B}\right)}{\alpha_{\rm K}^{(0)}}\left(\frac{\overline{B}}{\overline{B}_{\rm eq}}\right)^3, \tag{62}$$

and in a forced turbulence as

$$\frac{D_{\rm N}\left(\overline{B}\right)}{D_{\rm L}} \approx 32\left[1 + \frac{1}{8}\frac{D_{\rm cr}^{1/2}}{D_{\rm L}}\left(\frac{\ell_0}{L_B}\right)^2\left(\frac{\overline{B}}{\overline{B}_{\rm eq}}\right)\right]^{-2} \times \frac{\alpha\left(\overline{B}\right)}{\alpha_{\rm K}^{(0)}}\left(\frac{\overline{B}}{\overline{B}_{\rm eq}}\right)^3. \tag{63}$$

Equations (62)–(63) show that for the $\alpha^2\Omega$ dynamo, the nonlinear dynamo number decreases with increase of the mean magnetic field for a forced turbulence, and a shear-produced turbulence and a convective turbulence. This implies that the nonlinear mean-field $\alpha^2\Omega$ dynamo instability is always saturated for strong mean magnetic fields. When $(\zeta R_\omega)^2 \ll 1$, the poloidal and toroidal mean magnetic fields are of the same order of magnitude, so that Eqs. (62)–(63) do not contain factor $D_{\rm cr}^{1/2}/D_{\rm L}$, which is the ratio of energies of the poloidal and toroidal mean magnetic fields. This is similar to the mean-field nonlinear α^2 dynamo.

6 CONCLUSIONS

In the sun, stars and galaxies, the large-scale magnetic fields are originated due to the mean-field dynamo instabilities. The saturation of the dynamo generated large-scale magnetic fields is caused by algebraic and dynamic nonlinearities. However, these nonlinearities do not take into account the feedback of the generated mean magnetic field on the background turbulence. This nonlinear effect can be taking into account by means of the budget equation for the total turbulent energy. Using this approach and considering various origins of turbulence (e.g., a forced turbulence, a shear-produced turbulence and a convective turbulence), we have demonstrated that the mean-field $\alpha\Omega$, α^2 and $\alpha^2\Omega$ dynamo instabilities can be always saturated for any strong mean magnetic field. This is because the feedback of the generated mean magnetic field on the background turbulence in combination with the algebraic and dynamic nonlinearities, result in the decrease of the nonlinear dynamo number with increase of the mean magnetic field. These results have very important applications for astrophysical magnetic fields.

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DATA AVAILABILITY

There are no new data associated with this article.

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APPENDIX A: THE NONLINEAR FUNCTIONS

The nonlinear functions $q_{\rm p}(\beta)$ and $q_{\rm s}(\beta)$ are given by

$$q_{\rm p}(\beta) = \frac{2}{3\beta^2}\left[A_1^{(0)}(0) - A_1^{(0)}(\sqrt{2}\beta) - A_2^{(0)}(\sqrt{2}\beta)\right], \quad \text{(A1)}$$

$$q_{\rm s}(\beta) = -\frac{2}{3\beta^2}\, A_2^{(0)}(\sqrt{2}\beta), \quad \text{(A2)}$$

where $\beta = \sqrt{8}\ \overline{B}/\overline{B}_{\rm eq}$. For the derivation Eqs. (A1)–(A2) over the angles in $\mathbf{k}$-space we used the following identity:

$$\bar{I}_{ij} = \int \frac{k_{ij}\sin\theta}{1 + a\cos^2\theta}\, d\theta\, d\varphi = \bar{A}_1\delta_{ij} + \bar{A}_2\beta_{ij}, \quad \text{(A3)}$$

where $a = \beta^2/\bar{\tau}(k)$, and

$$\bar{A}_1 = \frac{2\pi}{a}\left[(a+1)\frac{\arctan(\sqrt{a})}{\sqrt{a}} - 1\right],$$

$$\bar{A}_2 = -\frac{2\pi}{a}\left[(a+3)\frac{\arctan(\sqrt{a})}{\sqrt{a}} - 3\right]$$

The functions $A_n^{(0)}(\beta)$ are given by

$$A_n^{(0)}(\beta) = \frac{3\beta^2}{\pi}\int_\beta^{\beta{\rm Rm}^{1/4}} \frac{\bar{A}_n(X^2)}{X^3}\, dX. \quad \text{(A4)}$$

The functions $A_1^{(0)}(\beta)$ and $A_2^{(0)}(\beta)$ are given by

$$A_1^{(0)}(\beta) = \frac{1}{5}\left[2 + 2\frac{\arctan\beta}{\beta^3}(3+5\beta^2) - \frac{6}{\beta^2} - \beta^2\ln{\rm Rm} - 2\beta^2\ln\left(\frac{1+\beta^2}{1+\beta^2\sqrt{\rm Rm}}\right)\right], \quad \text{(A5)}$$

$$A_2^{(0)}(\beta) = \frac{2}{5}\left[2 - \frac{\arctan\beta}{\beta^3}(9+5\beta^2) + \frac{9}{\beta^2} - \beta^2\ln{\rm Rm} - 2\beta^2\ln\left(\frac{1+\beta^2}{1+\beta^2\sqrt{\rm Rm}}\right)\right]. \quad \text{(A6)}$$

For $\overline{B} \ll \overline{B}_{\rm eq}/4{\rm Rm}^{1/4}$, these functions are given by

$$A_1^{(0)}(\beta) \sim 2 - \frac{1}{5}\beta^2\ln{\rm Rm},$$

$$A_2^{(0)}(\beta) \sim -\frac{2}{5}\beta^2\left[\ln{\rm Rm} + \frac{2}{15}\right].$$

For $\overline{B}_{\rm eq}/4{\rm Rm}^{1/4} \ll \overline{B} \ll \overline{B}_{\rm eq}/4$, these functions are given by

$$A_1^{(0)}(\beta) \sim 2 + \frac{2}{5}\beta^2\left[2\ln\beta - \frac{16}{15} + \frac{4}{7}\beta^2\right],$$

$$A_2^{(0)}(\beta) \sim \frac{2}{5}\beta^2\left[4\ln\beta - \frac{2}{15} - 3\beta^2\right],$$

and for $\overline{B} \gg \overline{B}_{\rm eq}/4$, they are given by

$$A_1^{(0)}(\beta) \sim \frac{\pi}{\beta} - \frac{3}{\beta^2}, \quad A_2^{(0)}(\beta) \sim -\frac{\pi}{\beta} + \frac{6}{\beta^2}.$$